

# Variational techniques in General Relativity: a metric affine approach to Kaluza's theory

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## Abstract

A new variational principle for General Relativity, based on an action functional  $I(\Phi, \nabla)$  involving both the metric  $\Phi$  and the connection  $\nabla$  as independent, *unconstrained* degrees of freedom is presented. The extremals of  $I$  are seen to be pairs  $(\Phi, \nabla)$  in which  $\Phi$  is a Ricci flat metric, and  $\nabla$  is the associated Riemannian connection. An application to Kaluza's theory of interacting gravitational and electromagnetic fields is discussed.

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## 1 Introduction

Several variational formulations of General Relativity, ranging from the purely metric approach of Hilbert and Einstein [1, 2] to Palatini's metric–affine formulation [3, 4, 5], to the more recent purely affine [6, 7, 8] purely frame [9] and frame–affine theories [10, 11, 12] have been so far proposed in the literature

In particular, in the metric–affine formulation, the dynamical fields are pairs  $(\Phi, \nabla)$  consisting of a pseudo–riemannian metric  $\Phi$  and of a torsionless linear connection  $\nabla$  on the space–time manifold  $\mathcal{V}_4$ . The corresponding variational principle relies on the action functional

$$I(\Phi, \nabla) = \int g^{ij} R_{ij} \sqrt{|g|} dx^1 \wedge \cdots \wedge dx^4$$

where  $g^{ij}$  are the contravariant components of the metric  $\Phi$ , and  $R_{ij} = R^h{}_{ihj}$  is the contracted curvature tensor associated with the connection  $\nabla$ . The stationarity

requirement for the functional  $I$  singles out extremal pairs  $(\Phi, \nabla)$  in which  $\Phi$  is a Ricci flat metric, and  $\nabla$  is the associated Riemannian connection.

In Palatini's approach, the absence of torsion, imposed as an a priori constraint, plays a crucial role in the deduction of the field equations (for a generalization of this viewpoint see e.g. [15]).

In this paper we propose an enhanced metric-affine principle, removing any restriction on the choice of the connection. In the resulting scheme both the absence of torsion and the condition  $\nabla\Phi = 0$  are part of the Euler-Lagrange equations associated with the action functional. The traditional Palatini-Hilbert and Einstein-Hilbert results are then recovered as special cases of the more general procedure.

As an application of the new geometrical setup, in § 3 we discuss a variational approach to Kaluza's theory of interacting electromagnetic and gravitational fields [13, 14]. The analysis relies on the introduction of a 5-dimensional principal fiber bundle  $M \rightarrow \mathcal{V}_4$  with structural group  $(\mathbb{R}, +)$ , accounting for the gauge-theoretical properties of the electromagnetic 4-potential. Following Kaluza, we then merge the gravitational and electromagnetic degrees of freedom into a symmetric tensor  $\hat{\Phi}$  of signature  $(4, 1)$ , playing the role of a metric tensor over  $M$ . We finally show that this metric, together with the associated Levi-Civita connection, are the extremals of a constrained variational problem of the proposed kind.

An advantage of the new formulation is that it involves only the physical (gravitational and electromagnetic) fields, and does not require any additional geometric object, such as the scalar field reported in [14].

## 2 The action principle

### 2.1 Mathematical preliminaries

Let  $M$  be an  $n$ -dimensional orientable manifold,  $\mathcal{R}(M) \xrightarrow{\pi} M$  the bundle of symmetric covariant tensors of rank 2 and signature  $(p, q)$  over  $M$ , and  $\mathcal{C}(M) \xrightarrow{\pi} M$  the bundle of linear connections over  $M$ .

The existence of *global* sections  $\Phi : M \rightarrow \mathcal{R}(M)$  is explicitly assumed. Each such section is called a *pseudo-riemannian metric* on  $M$ .

We refer  $M$  to local coordinates  $(U, x^1, \dots, x^n)$  and adopt a (possibly non holonomic) basis  $\{\partial_i, \omega^i, i = 1, \dots, n\}$  for the tensor algebra over  $U$ . The latter induces fiber coordinates on  $\mathcal{R}(M)$  and  $\mathcal{C}(M)$ , respectively denoted by  $x^i, y_{ij}$  and  $x^i, \gamma_{ij}^k$ .

The following results will be regarded as known:

- $\mathcal{C}(M) \xrightarrow{\pi} M$  is an affine bundle, modelled on the bundle  $\mathfrak{T}_2^1(M)$  of tensors contravariant of degree 1 and covariant of degree 2. In particular,  $\mathcal{C}(M)$  always admits *global* sections  $\nabla : M \rightarrow \mathcal{C}(M)$ . Each such section, locally represented as  $\gamma_{ij}^k = \Gamma_{ij}^k(x^1, \dots, x^n)$ , is called a *connection* over  $M$ . For any  $X \in D^1(M)$ , we denote by  $\nabla_X$  the *covariant derivative* along  $X$

induced by  $\nabla$ , namely the derivation of the tensor algebra  $\mathcal{D}(M)$  depending  $\mathcal{F}$ -linearly on  $X$  and commuting with contractions, uniquely determined by the requirement

$$\nabla_X(f) = X(f), \quad \nabla_{\partial_i} \partial_j = \Gamma_{ij}^k \partial_k \quad (2.1)$$

- $\mathcal{C}(M)$  carries an affine surjection  $T$ , known as the *torsion map*, into the subbundle of  $\mathfrak{T}_2^1(M)$  formed by the totality of tensors antisymmetric in the covariant indices. In local coordinates, denoting by  $C^i_{jk} := \langle [\partial_j, \partial_k], \omega^i \rangle$  the holonomy tensor of the basis  $\{\partial_i, \omega^i\}$ , we have the explicit representation  $T(\Gamma) = T^i_{jk}(\partial_i \otimes \omega^j \otimes \omega^k)_{\pi(\Gamma)}$ , with

$$T^i_{jk} = \gamma_{jk}^i - \gamma_{kj}^i - C^i_{jk} \quad (2.2)$$

- Assigning a pseudo-riemannian metric  $\Phi : M \rightarrow \mathcal{R}(M)$  singles out a distinguished section  $\hat{\nabla} : M \rightarrow \mathcal{C}(M)$ , called the *riemannian connection* of  $\Phi$ . The latter determines a bijection of  $\mathcal{C}(M)$  into the modelling space  $\mathfrak{T}_2^1(M)$  assigning to each  $\Gamma \in \mathcal{C}(M)$  the difference  $N(\Gamma) := \Gamma - \hat{\nabla}|_{\pi(\Gamma)}$ . Denoting by  $\hat{\Gamma}_{jk}^i(x^1, \dots, x^n)$  the connection coefficients of  $\hat{\nabla}$  in the basis  $\{\partial_i, \omega^i\}$ , the image  $N(\Gamma)$  is locally represented as  $N^i_{jk}(\partial_i \otimes \omega^j \otimes \omega^k)_{\pi(\Gamma)}$ , with

$$N^i_{jk} = \gamma_{jk}^i - \hat{\Gamma}_{jk}^i \quad (2.3)$$

In terms of  $N$ , eq. (2.2) provides the identification

$$T^i_{jk} = N^i_{jk} - N^i_{kj} \quad (2.4)$$

The fibered product  $\mathcal{R}(M) \times_M \mathcal{C}(M)$  is the natural environment for the development of a field theory in which every global section  $M \rightarrow \mathcal{R}(M) \times_M \mathcal{C}(M)$  corresponds to the simultaneous assignment of a pseudo-riemannian structure  $\Phi$  and of a connection  $\nabla$  over  $M$ . This is precisely the viewpoint we shall pursue. The field theory we shall discuss relies on the action functional

$$I(\phi, \nabla) := \int_D g^{ij} (R_{ij} + T_i T_j) \sqrt{|g|} \omega^1 \wedge \dots \wedge \omega^n \quad (2.5)$$

$R_{ij} := R^p_{ipj}$  and  $T_i := T^p_{pi}$  respectively denoting the contracted curvature tensor and the contracted torsion tensor of the connection  $\nabla$ . We shall prove that the extremals of the functional (2.5) are pairs  $(\Phi, \nabla)$  such that

- $\nabla$  is the riemannian connection of  $\Phi$ ;
- the metric  $\Phi$  is “Ricci flat”, i.e. it obeys Einstein’s equation *in vacuo*

$$R_{ij} = 0$$

## 2.2 The field equations

To fulfill our program, we refer  $\mathcal{R}(M) \times_M \mathcal{C}(M)$  to coordinates  $x^i, y_{ij}, \gamma_{ij}^k$ . Every section  $(\Phi, \nabla) : M \rightarrow \mathcal{R}(M) \times_M \mathcal{C}(M)$  is then described locally as

$$y_{ij} = g_{ij}(x^1, \dots, x^n), \quad \gamma_{ij}^k = \Gamma_{ij}^k(x^1, \dots, x^n) \quad (2.6)$$

We denote by  $\omega^i_j := \Gamma_{kj}^i \omega^k$  the connection 1-forms of  $\nabla$  in the basis  $\{\partial_i, \omega^i\}$ , and by  $\theta^i := \frac{1}{2} T^i_{jk} \omega^j \wedge \omega^k$  and  $\rho^i_j := \frac{1}{2} R^i_{jkl} \omega^k \wedge \omega^l$  respectively the torsion 2-forms and the curvature 2-forms of  $\nabla$ . The relationships between the various objects are summarized into Cartan's structural equations

$$\theta^i = d\omega^i + \omega^i_p \wedge \omega^p \quad (2.7a)$$

(pointwise equivalent to eq. (2.2)) and

$$\rho^i_j = d\omega^i_j + \omega^i_p \wedge \omega^p_j \quad (2.7b)$$

In terms of  $\theta^i$  and  $\rho^i_j$ , the contracted torsion and curvature tensors involved in eq. (2.5) are respectively expressed by the relations

$$T_i = \langle \partial_p \wedge \partial_i | \theta^p \rangle, \quad R_{ij} = \langle \partial_p \wedge \partial_j | \rho^p_i \rangle \quad (2.8)$$

We keep the notation  $\hat{\nabla}$  for the Riemannian connection of  $\Phi$ , and denote by a hat all quantities pertaining to  $\hat{\nabla}$  (connection coefficients, connection 1-forms, etc.).

According to eq. (2.3), the relation between the connection 1-forms of  $\nabla$  and those of  $\hat{\nabla}$  is locally expressed as

$$\omega^i_j = \hat{\omega}^i_j + N^i_j \quad (2.9)$$

with  $N^i_j := N^i_{kj} \omega^k = (\Gamma_{kj}^i - \hat{\Gamma}_{kj}^i) \omega^k$ .

On account of eqs. (2.7a, b), this yields the identifications

$$T_i = \langle \partial_p \wedge \partial_i | N^p_q \omega^q \rangle = \delta_{pi}^{rq} N^p_{rq} \quad (2.10a)$$

$$R_{ij} = \langle \partial_p \wedge \partial_j | \hat{\rho}^p_i + dN^p_i + N^p_q \wedge \hat{\omega}^q_i + \hat{\omega}^p_q \wedge N^q_i + N^p_q \wedge N^q_i \rangle \quad (2.10b)$$

On the other hand, a straightforward computation provides the relation

$$dN^p_i + N^p_q \wedge \hat{\omega}^q_i + \hat{\omega}^p_q \wedge N^q_i = \hat{\nabla}_{\partial_k} N^p_{ri} \omega^k \wedge \omega^r$$

Collecting all results, we end up with the expression

$$\begin{aligned} g^{ij} (R_{ij} + T_i T_j) &= g^{ij} \left[ \hat{R}_{ij} + \delta_{pj}^{kr} \left( \hat{\nabla}_{\partial_k} N^p_{ri} + N^p_{kq} N^q_{ri} \right) + T_i T_j \right] = \\ &= g^{ij} \left( \hat{R}_{ij} + \delta_{pj}^{kr} N^p_{kq} N^q_{ri} + T_i T_j \right) + g^{ij} \left( \hat{\nabla}_{\partial_p} N^p_{ji} - \hat{\nabla}_{\partial_j} N^p_{pi} \right) \end{aligned} \quad (2.11)$$

This shows that, up to a divergence, the action functional (2.5) may be written in the equivalent form

$$I(\Phi, \nabla) = \int_D g^{ij} \left( \hat{R}_{ij} + \delta_{pj}^{kr} N^p{}_{kq} N^q{}_{ri} + T_i T_j \right) \sqrt{|g|} \omega^1 \wedge \dots \wedge \omega^n \quad (2.12)$$

with  $T_i$  given by eq. (2.10a) and with  $\hat{R}_{ij} = \hat{R}^p{}_{ipj}$  representing the *Ricci tensor* associated with the metric  $\Phi$ .

Both expressions (2.5), (2.12) have their own advantages: eq. (2.5) depends algebraically on  $\Phi$ , thereby allowing a simple description of the variation of the functional  $I$  under arbitrary deformations of the metric. On the contrary, eq. (2.12) depends algebraically on  $\nabla$ , thus yielding an equally simple expression for  $\delta I$  under arbitrary deformations of the connection. Let us work out both aspects in detail.

1) On account of the relation

$$\frac{\partial \sqrt{|g|}}{\partial g^{ab}} = \frac{\partial}{\partial g^{ab}} \frac{1}{\sqrt{|g|^{-1}}} = -\frac{1}{2} |g|^{\frac{3}{2}} \frac{\partial g^{-1}}{\partial g^{ab}} = -\frac{1}{2} \sqrt{|g|} g_{ab} \quad (2.13)$$

the variation of  $I$  under arbitrary deformations  $\delta g^{ab}$  takes the form

$$\delta I = \int_D \left[ R_{ab} + T_a T_b - \frac{1}{2} (R + T_p T^p) g_{ab} \right] \delta g^{ab} \sqrt{|g|} \omega^1 \wedge \dots \wedge \omega^n \quad (2.14)$$

with  $R = g^{ab} R_{ab}$  and  $T^p = g^{pq} T_q$ .

In the case of *unconstrained* deformations, the requirement  $\delta I = 0$  is therefore expressed by the condition

$$R_{ab} + T_a T_b - \frac{1}{2} (R + T_p T^p) g_{ab} = 0 \quad (2.15)$$

In dimension  $n > 2$  the latter reduces to

$$R_{ab} + T_a T_b = 0 \quad (2.16)$$

If the class admissible metrics is restricted to a subfamily  $\Phi(\xi^1, \dots, \xi^r)$  controlled by a smaller number of fields, eq. (2.14) is still valid, but eq. (2.15) is replaced by the system

$$\left[ R_{ab} + T_a T_b - \frac{1}{2} (R + T_p T^p) g_{ab} \right] \frac{\partial g^{ab}}{\partial \xi^j} = 0 \quad (2.17)$$

An example of this situation will be illustrated in Section 3.

2) In order to evaluate the variation  $\delta I$  under arbitrary deformations of the connection we resort to the representation (2.12). From the latter, making use of the identifications  $\delta \Gamma_{bc}{}^a = \delta N^a{}_{bc}$ ,  $\delta T_i = \delta_{ai}^{bc} \delta N^a{}_{bc}$  we get the expression

$$\delta I = \int_D \left[ \delta_{pj}^{kr} \left( \delta_a^p \delta_k^b N^c{}_{rj} + \delta_r^b g^{jc} N^p{}_{ka} \right) + 2 \delta_{ai}^{bc} T^i \right] \delta N^a{}_{bc} \sqrt{|g|} \omega^1 \wedge \dots \wedge \omega^n \quad (2.18)$$

In the case of unconstrained deformations  $\delta N^a_{bc}$ , the requirement  $\delta I = 0$  is therefore expressed by the condition

$$0 = \delta_{aj}^{br} N^c_r{}^j + \delta_{pj}^{kb} N^p_{ka} g^{jc} + 2\delta_{ai}^{bc} T^i \quad (2.19)$$

From the latter, contracting  $a$  with  $c$ , we derive the relation

$$0 = -2(n-1)T^b \quad \implies \quad T^b = 0 \quad (2.20)$$

In view of this, eq. (2.19) reduces to

$$0 = \delta_a^b N^c_r{}^r - N^c_a{}^b + N^r_{ra} g^{bc} - N^{bc}_a \quad (2.21)$$

Setting  $X^c := N^c_r{}^r$ ,  $Y_a := N^r_{ra}$  and lowering all indices, eq. (2.21) takes the form

$$N_{cab} + N_{bca} = g_{ab} X_c + g_{bc} Y_a$$

The latter is easily solved for  $N_{cab}$ , yielding the expression

$$2N_{cab} = g_{ab}(X_c - Y_c) + g_{ac}(Y_b - X_b) + g_{bc}(Y_a + X_a)$$

From this, recalling eqs. (2.10a), (2.20) as well as the definition of  $Y_a$ , we get the relations

$$\begin{aligned} 2Y_b &= 2g^{ac} N_{cab} = X_b + n(Y_b - X_b) + X_b &\Rightarrow & (n-2)(Y_b - X_b) = 0 \\ 0 &= T_a = g^{bc}(N_{cab} - N_{cba}) = (n+1)Y_a &\Rightarrow & Y_a = 0 \end{aligned}$$

Collecting all results we conclude that, for  $n > 2$ , the requirement  $\delta I = 0$  is mathematically equivalent to  $N^i_{jk} = 0$ , i.e. to the identification  $\nabla = \hat{\nabla}$ .

This fact, together with eq. (2.16), provides a full proof of the result stated in § 2.1.

**Remark 2.1** Since being an extremal with respect to a class  $\mathfrak{E}$  of deformations automatically implies being an extremal with respect to any subclass  $\mathfrak{E}' \subset \mathfrak{E}$ , the consequences of the variational principle based on the functional (2.5) hold unchanged if part of the conditions arising from the requirement  $\nabla = \hat{\nabla}$  are imposed as a priori constraints. Thus, for example, if the choice of  $\nabla$  is restricted to the class of torsionless connections, the previous analysis provides a proof of the Palatini–Hilbert action principle.

More radically, if one gives up the affine degrees of freedom and considers a purely metric setup, with the ansatz  $\nabla = \hat{\nabla}$  imposed at the outset, the action principle (2.5) is easily recognized to yield back the Einstein–Hilbert one.

### 3 Affine scalars and the Einstein–Maxwell theory

As an illustration of the results developed so far we discuss an application of the functional (2.5) to the study of the Einstein–Maxwell equations. The argument provides a geometric approach to Kaluza’s theory of interacting gravitational and electromagnetic fields, free of any spurious, non–physical field (see e.g. [14] and references therein).

Let  $\mathcal{V}_4$  denote a 4–dimensional orientable space–time manifold, admitting a pseudo–riemannian structure of signature (3,1). Also, let  $M \xrightarrow{\pi} \mathcal{V}_4$  denote a principal fiber bundle over  $\mathcal{V}_4$  with structural group  $(\mathbb{R}, +)$ , henceforth referred to as the bundle of *affine scalars*.

The bundle  $M \rightarrow \mathcal{V}_4$  is globally trivial. Assigning a trivialization  $u : M \rightarrow \mathbb{R}$  allows to lift every coordinate system  $x^1, \dots, x^4$  in  $\mathcal{V}_4$  to a corresponding fibered coordinate system  $u, x^1, \dots, x^4$  in  $M$ . The group of fibered coordinate transformations has then the form

$$\bar{u} = u + f(x^1, \dots, x^4), \quad \bar{x}^i = x^i(x^1, \dots, x^4) \quad (3.1)$$

In fibered coordinates, the generator of the action of  $(\mathbb{R}, +)$ , commonly referred to as the *fundamental vector field* of  $M$ , coincides with the field  $\partial_u := \frac{\partial}{\partial u}$ .

The presence of  $\partial_u$  singles out a distinguished sub–bundle  $\hat{\mathcal{R}}(M) \rightarrow M$  of the bundle of pseudo–riemannian structures of signature (4,1) over  $M$ , formed by the totality of metrics satisfying the condition  $(\partial_u, \partial_u) = 1$ .

Through an obvious composition of maps,  $\hat{\mathcal{R}}(M)$  may be viewed as a fiber bundle over  $\mathcal{V}_4$ . In the resulting context, assigning a section  $\Upsilon : \mathcal{V}_4 \rightarrow \hat{\mathcal{R}}(M)$  is then equivalent to assigning a pair  $(\psi, \hat{\Phi})$  where

- $\psi : \mathcal{V}_4 \rightarrow M$  is a section, described locally as  $u = \psi(x^1, \dots, x^4)$ ;
- $\hat{\Phi} : M \rightarrow \hat{\mathcal{R}}(M)$  is a pseudo–riemannian metric on  $M$ , uniquely characterized by the requirements

$$\hat{\Phi}|_{\psi(x)} = \Upsilon(x) \quad \forall x \in \mathcal{V}_4, \quad \mathcal{L}_{\partial_u} \hat{\Phi} = 0 \quad (3.2a)$$

locally summarized into the representation

$$\hat{\Phi} = du \otimes du + 2\gamma_i(x^1, \dots, x^4) du \odot dx^i + \gamma_{ij}(x^1, \dots, x^4) dx^i \otimes dx^j \quad (3.2b)$$

Denoting by the  $g : T(M) \rightarrow T^*(M)$  the process of “lowering the indices” induced by the metric (3.2b), let us now consider the 1–form

$$\sigma := g(\partial_u) = du + \gamma_i dx^i \quad (3.3a)$$

By direct computation we have then the relations  $\langle \sigma, \partial_u \rangle = 1$ ,  $\mathcal{L}_{\partial_u} \sigma = 0$ , indicating that  $\sigma$  defines a *principal connection* relative to the fibration  $M \rightarrow \mathcal{V}_4$ . For each choice of the section  $\psi$ , the knowledge of  $\sigma$  is therefore equivalent to the knowledge of the pull–back  $\psi^*(\sigma) \in \mathcal{D}_1(\mathcal{V}_4)$ .

In a similar way, in view of eqs. (3.2b), (3.3a), the difference  $\hat{\Phi} - \sigma \otimes \sigma$  is easily recognized to coincide with the pull-back of a tensor field over  $\mathcal{V}_4$ , expressed in coordinates as

$$\Phi = (\gamma_{ij} - \gamma_i \gamma_j) dx^i \otimes dx^j := g_{ij} dx^i \otimes dx^j \quad (3.3b)$$

Collecting all results, we have therefore the representation

$$\hat{\Phi} = \sigma \otimes \sigma + \pi^*(\Phi) \quad (3.4)$$

The idea is now to interpret the tensor (3.3b) as a pseudo-riemannian metric on  $\mathcal{V}_4$ , accounting for the gravitational effects, and to regard the connection  $\sigma$  as a description of the electromagnetic field. In this way, up to a dimensional constant  $\kappa$  depending on the choice of the units, every section  $\Upsilon : \mathcal{V}_4 \rightarrow \hat{\mathcal{R}}(M)$  is identified with a triple  $(\psi, A, \Phi)$  where:

- $\psi : \mathcal{V}_4 \rightarrow M$  is a section, accounting for the electromagnetic gauge;
- $A := \kappa^{-1} \psi^*(\sigma)$  is the ( $\psi$ -dependent) electromagnetic 4-potential in  $\mathcal{V}_4$ ;
- $\Phi = g_{ij} dx^i \otimes dx^j$  is the ( $\psi$ -independent) metric tensor of  $\mathcal{V}_4$ .

The algorithm is significantly simplified referring the tensor algebra  $\mathcal{D}(M)$  to the local non-holonomic basis  $\{\partial_A, \omega^A, A = 0, \dots, 4\}$  defined by the ansatz

$$\omega^0 = \sigma, \quad \omega^i = dx^i, \quad \partial_0 = \partial_u = \frac{\partial}{\partial u}, \quad \partial_i = \frac{\partial}{\partial x^i} - \gamma_i \frac{\partial}{\partial u} \quad (3.5)$$

In view of eqs. (3.4), (3.5), the covariant representation of the metric reads

$$\hat{\Phi} := g_{AB} \omega^A \otimes \omega^B = \omega^0 \otimes \omega^0 + g_{ij} \omega^i \otimes \omega^j \quad (3.6a)$$

In a similar way, the contravariant representation of  $\hat{\Phi}$  takes the form

$$g^{AB} \partial_A \otimes \partial_B = g^{ij} \partial_i \otimes \partial_j + \partial_0 \otimes \partial_0 \quad (3.6b)$$

with  $g^{ij} g_{jk} = \delta_k^i$ . Eqs. (3.6a, b) imply the identity

$$1 = g^{00} = \frac{\det g_{ij}}{\det g_{AB}} \quad (3.7)$$

Setting  $\det g_{AB} = \det g_{ij} := g$ , and denoting by  $\hat{\varepsilon} := \sqrt{|g|} \omega^0 \wedge \dots \wedge \omega^4$  and  $\varepsilon := \sqrt{|g|} dx^1 \wedge \dots \wedge dx^4$  the Ricci tensors respectively associated with the metrics  $\hat{\Phi}$  and  $\Phi$ , eqs. (3.3a), (3.5), (3.7a) yield the relations

$$\hat{\varepsilon} = \omega^0 \wedge \pi^*(\varepsilon) = \sqrt{|g|} du \wedge dx^1 \wedge \dots \wedge dx^4 \quad (3.8a)$$

$$\partial_0 \lrcorner \hat{\varepsilon} = \pi^*(\varepsilon) = \sqrt{|g|} dx^1 \wedge \dots \wedge dx^4 \quad (3.8b)$$

Given any section  $\Upsilon = (\psi, \hat{\Phi}) : \mathcal{V}_4 \rightarrow \hat{\mathcal{R}}(M)$ , let  $\hat{\nabla}$  denote the riemannian connection of  $\hat{\Phi}$ . The 4+1 decomposition expressed by eq. (3.4) is then reflected



into an analogous representation of  $\hat{\nabla}$  in terms of the Christoffel symbols of  $\Phi$  and of the curvature 2-form of  $\sigma$ . Setting

$$\hat{\omega}^A{}_B := \hat{\Gamma}_{CB}{}^A \omega^C, \quad \omega^{*i}{}_j := \left\{ \begin{matrix} i \\ k j \end{matrix} \right\} \omega^k, \quad \Omega := d\omega^0 = \Omega_{ij} \omega^i \wedge \omega^j \quad (3.9a)$$

a straightforward calculation yields the results

$$\hat{\omega}^i{}_j = \omega^{*i}{}_j - \Omega^i{}_j \omega^0, \quad \hat{\omega}^i{}_0 = -\Omega^i{}_j \omega^j, \quad \hat{\omega}^0{}_i = \Omega_{ij} \omega^j, \quad \hat{\omega}^0{}_0 = 0 \quad (3.9b)$$

To complete our geometrical setup let us finally denote by  $\hat{\mathcal{R}}(M) \times_M \mathcal{C}(M)$  the fibered product of  $\hat{\mathcal{R}}(M)$  with the bundle of linear connections over  $M$ , and by  $p_1 : \hat{\mathcal{R}}(M) \times_M \mathcal{C}(M) \rightarrow \hat{\mathcal{R}}(M)$ ,  $p_2 : \hat{\mathcal{R}}(M) \times_M \mathcal{C}(M) \rightarrow \mathcal{C}(M)$  the associated natural projections. Once again, we regard  $\hat{\mathcal{R}}(M) \times_M \mathcal{C}(M)$  as a fiber bundle over  $\mathcal{V}_4$ . Assigning a section  $\Xi : \mathcal{V}_4 \rightarrow \hat{\mathcal{R}}(M) \times_M \mathcal{C}(M)$  is then easily recognized to be mathematically equivalent to assigning a triple  $(\psi, \hat{\Phi}, \nabla)$ , where:

- the pair  $(\psi, \hat{\Phi})$  is defined exactly as above, with the section  $\Upsilon : \mathcal{V}_4 \rightarrow \hat{R}(M)$  now identified with the product  $p_1 \cdot \Xi$ ;
- $\nabla : M \rightarrow \mathcal{C}(M)$  is a linear connection over  $M$ , invariant under the action of the structural group  $(\mathbb{R}, +)$  and satisfying  $\nabla|_z = p_2 \cdot \Xi(\pi(z)) \forall z \in \psi(\mathcal{V}_4)$ . As intuitively clear this means that, in the non-holonomic basis (3.5),  $\nabla$  is described by connection coefficients  $\Gamma_{AB}{}^C$  *independent of the variable  $u$* <sup>1</sup>.

After these preliminaries, let us now adapt the variational scheme of § 2 to context in study. To this end, to every section  $\Xi : \mathcal{V}_4 \rightarrow \hat{\mathcal{R}}(M) \times_M \mathcal{C}(M)$ , viewed as a triple  $(\psi, \hat{\Phi}, \nabla)$  in the sense described above, we associate the action functional

$$I(\Xi) := \int_{\psi(D)} g^{AB} (R_{AB} + T_A T_B) \sqrt{|g|} dx^1 \wedge \cdots \wedge dx^4 \quad (3.10)$$

$D$  being any domain with compact closure in  $\mathcal{V}_4$ , and  $\sqrt{|g|} dx^1 \wedge \cdots \wedge dx^4$  denoting the invariant 4-form (3.8b).

Up to straightforward notational changes, evaluating of the right hand side of eq. (3.10) involves the same type of algorithm already exploited in § 2. In particular eq. (2.11) takes now the form

$$\begin{aligned} g^{AB} (R_{AB} + T_A T_B) &= g^{AB} \left( \hat{R}_{AB} + \delta_{PB}^{HK} N^P{}_{HQ} N^Q{}_{KA} + T_A T_B \right) + \\ &\quad + \hat{\nabla}_{\partial_A} \left( N^A{}_{P^P} - N^P{}_{P^A} \right) \end{aligned} \quad (3.11)$$

The last term in eq. (3.11) is the divergence of a vector field  $X$  on  $M$ , with components  $X^A := N^P{}_{P^A} - N^A{}_{P^P}$  independent of the variable  $u$ .

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<sup>1</sup>A review of the concept of Lie derivative of a connection is reported in Appendix A.

In coordinates, recalling eqs. (3.8a, b) this implies the exactness relation

$$\hat{\nabla}_{\partial_A} X^A \sqrt{|g|} dx^1 \wedge \cdots \wedge dx^4 = \frac{\partial}{\partial x^i} \left( \sqrt{|g|} X^i \right) dx^1 \wedge \cdots \wedge dx^4 = d(X \lrcorner \pi^*(\varepsilon))$$

Once again, up to unessential contributions, we are thus left with the expression

$$I(\Xi) = \int_{\psi(D)} g^{AB} \left( \hat{R}_{AB} + \delta_{PB}^{HK} N^P{}_{HQ} N^Q{}_{KA} + T_A T_B \right) \sqrt{|g|} dx^1 \wedge \cdots \wedge dx^4 \quad (3.12)$$

Due to this fact, the analysis of the action principle  $\delta I = 0$  may be carried on along the same lines illustrated in § 2. Partly from this and partly by inspection of eqs. (3.10), (3.12), we derive the following conclusions:

- the value of the functional  $I$  is invariant under arbitrary deformations of the section  $\psi$ . Therefore, the requirement  $\delta I = 0$  does not pose any condition on the choice of  $\psi$ , consistently with the interpretation of the latter as a *gauge field*;
- the variation of the right hand side of eq. (3.12) under arbitrary deformations of the components  $N^A{}_{BC}$  takes the form (2.18), with all indices written in uppercase. The requirement  $\delta I = 0$  is therefore equivalent to the condition  $N^A{}_{BC} = 0$ , i.e. to the identification  $\nabla = \hat{\nabla}$ ;
- in order to express the deformation of the metric in the non-holonomic basis  $\{\partial_A, \omega^A\}$  care must be taken of the fact that the basis itself gets modified by the deformation. To account for this fact, we start with the representation (3.6b). From the latter we get the relation

$$\delta(g^{AB} \partial_A \otimes \partial_B) = \delta g^{ij} \partial_i \otimes \partial_j + g^{ij} (\delta \partial_i \otimes \partial_j + \partial_i \otimes \delta \partial_j)$$

whence, setting  $\delta(g^{AB} \partial_A \otimes \partial_B) := \delta \hat{\phi}^{AB} \partial_A \otimes \partial_B$  and recalling eq. (3.5)

$$\delta \hat{\phi}^{AB} = \left\langle \delta g^{ij} \partial_i \otimes \partial_j - g^{ij} (\delta \gamma_i \partial_u \otimes \partial_j + \delta \gamma_j \partial_i \otimes \partial_u), \omega^A \otimes \omega^B \right\rangle \quad (3.13)$$

Comparison of eq. (3.13) with eqs. (2.13), (3.6) provides the further identification

$$\delta \sqrt{|g|} = -\frac{1}{2} g_{ij} \delta g^{ij} = -\frac{1}{2} g_{AB} \delta \hat{\phi}^{AB}$$

The variation of the functional  $I$  under admissible deformations of the metric takes therefore the form

$$\delta I = \int_{\psi(D)} \left[ R_{AB} + T_A T_B - \frac{1}{2} (R + T_P T^P) g_{AB} \right] \delta \hat{\Phi}^{AB} \sqrt{|g|} dx^1 \wedge \cdots \wedge dx^4$$

with  $\delta \hat{\phi}^{AB}$  given by eq. (3.13). Collecting all results we conclude

**Proposition 3.1** *A necessary condition for a section  $\Xi : \mathcal{V}_4 \rightarrow \hat{\mathcal{R}}(M) \times_M \mathcal{C}(M)$  to be an extremal for the functional (3.10) under arbitrary deformations of all fields  $\psi, \hat{\Phi}, \nabla$  is the validity of the relations*

$$\nabla = \hat{\nabla} \quad (3.14a)$$

$$\hat{R}_{0j} = \hat{R}_{j0} = 0 \quad (3.14b)$$

$$\hat{R}_{ij} - \frac{1}{2} \left( \hat{R}^0_0 + \hat{R}^k_k \right) g_{ij} = 0 \quad (3.14c)$$

As a final step we now rephrase eqs. (3.14b, c) in terms of the physical fields, namely the metric of  $\mathcal{V}_4$  and the electromagnetic tensor  $F := d(\kappa^{-1} \psi^*(\omega^0)) = \kappa^{-1} \Omega_{ij} \omega^i \wedge \omega^j$ . To this end, we evaluate the curvature 2-forms of  $\hat{\nabla}$  in terms of  $\Omega^i_j$  and of the curvature 2-forms  $\rho^{*i}_j$  of the riemannian connection over  $\mathcal{V}_4$ . On account of eqs. (3.5), (3.9a, b), a straightforward calculation yields the result

$$\begin{aligned} \hat{\rho}^i_j &= d\hat{\omega}^i_j + \hat{\omega}^i_r \wedge \hat{\omega}^r_j + \hat{\omega}^i_0 \wedge \hat{\omega}^0_j = \\ &= \rho^{*i}_j - (\Omega^i_j \Omega_{rs} + \Omega^i_r \Omega_{js}) \omega^r \wedge \omega^s - (d\Omega^i_j + \omega^{*i}_p \Omega^p_j - \omega^{*p}_j \Omega^i_p) \wedge \omega^0 \\ \hat{\rho}^0_j &= d\hat{\omega}^0_j + \hat{\omega}^0_r \wedge \hat{\omega}^r_j = (d\Omega_{jr} - \omega^{*k}_j \Omega_{kr}) \wedge \omega^r - \Omega^k_j \Omega_{kr} \omega^r \wedge \omega^0 \\ \hat{\rho}^j_0 &= d\hat{\omega}^j_0 + \hat{\omega}^j_r \wedge \hat{\omega}^r_0 = -(d\Omega^j_r - \omega^{*j}_k \Omega^k_r) \wedge \omega^r - \Omega^j_k \Omega^k_r \omega^r \wedge \omega^0 \\ \hat{\rho}^0_0 &= \hat{\omega}^0_r \wedge \hat{\omega}^r_0 = -\Omega_{pr} \Omega^p_s \omega^r \wedge \omega^s \end{aligned}$$

From this, resuming the standard notation of tensor calculus on  $\mathcal{V}_4$  ( $R_{ij}$  for the Ricci tensor,  $w^i_{j\dots h\parallel k}$  for the covariant derivative, etc.), and recalling eq. (2.8) we get the identifications

$$\hat{R}_{ij} = R_{ij} - 2\Omega^p_i \Omega_{pj}, \quad \hat{R}_{i0} = \hat{R}_{0i} = \Omega_i^p{}_{\parallel p}, \quad \hat{R}_{00} = \Omega^{rs} \Omega_{rs} \quad (3.15)$$

Collecting all results, and writing  $\kappa F_{ij}$  in place of  $\Omega_{ij}$  we conclude that, with the ansatz  $\kappa = \frac{\sqrt{4\pi G}}{c^2}$ , eqs. (3.14) are identical to the Einstein Maxwell equations

$$\begin{cases} F_i{}^p{}_{\parallel p} = 0 \\ R_{ij} - \frac{1}{2} R g_{ij} = \frac{8\pi G}{c^4} \left( F^p_i F_{pj} - \frac{1}{4} F^{rs} F_{rs} g_{ij} \right) \end{cases}$$

Once again, it is worth remarking that all previous conclusions hold unchanged if part of the relations expressed by the Euler-Lagrange equations (3.14a, b, c) are imposed as a priori constraints. In particular, if the requirement  $\nabla = \hat{\nabla}$  is assumed at the outset — thus giving up the affine degrees of freedom and regarding the dynamical fields as sections  $\Upsilon : \mathcal{V}_4 \rightarrow \mathcal{R}(M)$  in the sense illustrated at the beginning of this Section — the functional (3.10) reduces to

$$I = \int_{\psi(D)} g^{AB} \hat{R}_{AB} \sqrt{|g|} dx^1 \wedge \dots \wedge dx^4$$

Recalling eqs. (3.6b), (3.15), and evaluating everything in terms of the physical fields  $\Phi$  and  $F = \kappa^{-1}\Omega$ , the latter expression may be written in the form

$$\begin{aligned} I &= \int_D \psi^* \left( g^{AB} \hat{R}_{AB} \right) \sqrt{|g|} dx^1 \wedge \cdots \wedge dx^4 = \\ &= \int_D \left[ R - \frac{4\pi G}{c^4} F_{ij} F^{ij} \right] \sqrt{|g|} dx^1 \wedge \cdots \wedge dx^4 \end{aligned} \quad (3.16)$$

Under the stated circumstance, the requirement  $\delta I = 0$  is therefore identical to the action principle for the Einstein–Maxwell equations in General Relativity.

## A Lie derivative of connections

Let  $M$  be an  $n$ -dimensional differentiable manifold. We denote by  $L(M) \xrightarrow{\pi} M$  the frame bundle of  $M$ , and by  $r_\alpha(\zeta) = \zeta \cdot \alpha$  the right action of  $GL(n, \mathbb{R})$  on  $L(M)$ . Given any local chart  $(U, x^1, \dots, x^n)$  in  $M$ , we refer  $L(M)$  to fiber coordinates  $x^i, y^i_j$  according to the prescriptions

$$x^i(\zeta) = x^i(\pi(\zeta)), \quad \zeta_i = \left( \frac{\partial}{\partial x^j} \right)_{\pi(\zeta)} y^j_i(\zeta) \quad \forall \zeta = \{\zeta_1, \dots, \zeta_n\} \in \pi^{-1}(U)$$

In these coordinates, the Lie algebra associated with the action of  $GL(n, \mathbb{R})$  on  $L(M)$  is spanned by the vector fields

$$\mathfrak{X}_p^j = y^i_p \frac{\partial}{\partial y^i_j} \quad (A.1)$$

commonly referred to as the *fundamental vector fields* of  $L(M)$ .

A vector valued 1-form  $\lambda^i_{j\dots k} = \lambda^i_{j\dots k r} dx^r + \lambda^i_{j\dots k a} dy^a_b$  over  $L(M)$  is called *pseudo-tensorial* if and only if it obeys the transport law [16]

$$r_\alpha^*(\lambda^i_{j\dots k}) = (\alpha^{-1})^i_p \alpha^q_j \cdots \alpha^r_k \lambda^p_{q\dots r} \quad \forall \alpha \in GL(n, \mathbb{R}) \quad (A.2)$$

The definition is immediately extended to vector valued  $r$ -forms. A semibasic pseudo-tensorial  $r$ -form is called *tensorial*. The reason for this denomination is that, given any tensorial  $r$ -form  $\lambda^i_{j\dots k}$ , the vector valued function

$$\lambda^i_{j\dots k b_1\dots b_r} := y^{a_1}_{b_1} \cdots y^{a_r}_{b_r} \left\langle \lambda^i_{j\dots k} \left| \frac{\partial}{\partial x^{a_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{a_r}} \right. \right\rangle$$

defines a tensor field over  $M$ , whose components in any basis  $\zeta$  coincide with the values  $\lambda^i_{j\dots k b_1\dots b_r}(\zeta)$ .

Every vector field  $X \in \mathcal{D}^1(M)$  may be lifted to a field  $\tilde{X} \in \mathcal{D}^1(L(M))$ , related in an obvious way to the push forward of the 1-parameter group of diffeomorphisms induced by  $X$ . The operation, described in coordinates as

$$X = X^i \frac{\partial}{\partial x^i} \quad \longrightarrow \quad \tilde{X} = X^i \frac{\partial}{\partial x^i} + \frac{\partial X^i}{\partial x^k} y^k_j \frac{\partial}{\partial y^i_j} \quad (A.3)$$

is called the *universal lift* of vector fields. By construction, the field  $\tilde{X}$  is invariant under the action of  $GL(n, \mathbb{R})$ , as confirmed by the commutation relations

$$\left[ \tilde{X}, \mathfrak{X}_p^j \right] = \left[ X^r \frac{\partial}{\partial x^r} + \frac{\partial X^r}{\partial x^k} y^k_s \frac{\partial}{\partial y^r_s}, y^i_p \frac{\partial}{\partial y^i_j} \right] = 0 \quad (\text{A.4})$$

Due to this fact, the 1-parameter group of diffeomorphisms associated with  $\tilde{X}$  *commutes* with the action of  $GL(n, \mathbb{R})$ . Given any pseudo-tensorial  $r$ -form  $\lambda^{i_j \dots k}$ , the Lie derivative  $\mathcal{L}_{\tilde{X}} \lambda^{i_j \dots k}$  is therefore once again pseudo-tensorial.

By definition, a connection  $\nabla : M \rightarrow \mathcal{C}(M)$ , locally described by connection 1-forms  $\omega^k_j := \Gamma_{ij}^k dx^i$  is a *horizontal distribution* in  $L(M)$ , identified with the annihilator of the vector-valued pseudo-tensorial 1-form [16]

$$\tilde{\omega}^a_b = (y^{-1})^a_r (dy^r_b + \omega^r_s y^s_b) \quad (\text{A.5})$$

In view of our previous observations, given any vector field  $X$  on  $M$ , the Lie derivative  $\mathcal{L}_{\tilde{X}} \tilde{\omega}^a_b$  along the universal lift of  $X$  is then a pseudo-tensorial 1-form over  $L(M)$ . Moreover, eqs. (A.2)–(A.4) imply the relation

$$\left\langle \mathcal{L}_{\tilde{X}} \tilde{\omega}^a_b, \mathfrak{X}_p^j \right\rangle = \tilde{X} \left( \left\langle \tilde{\omega}^a_b, \mathfrak{X}_p^j \right\rangle \right) - \left\langle \tilde{\omega}^a_b, \left[ \tilde{X}, \mathfrak{X}_p^j \right] \right\rangle = \tilde{X} \left( \delta_p^a \delta_p^j \right) = 0$$

showing that  $\mathcal{L}_{\tilde{X}} \tilde{\omega}^a_b$  is also semibasic, and has therefore a *tensorial* character. As such,  $\mathcal{L}_{\tilde{X}} \tilde{\omega}^a_b$  defines a tensor field of type  $(1, 2)$  over  $M$ , henceforth denoted by  $\mathcal{L}_X \nabla$ , and called the *Lie derivative* of the connection  $\nabla$  along  $X$ .

In particular, if the local coordinates are chosen consistently with the requirement  $X = \frac{\partial}{\partial x^1}$ , eqs. (A.3), (A.5) provide the relation

$$\mathcal{L}_{\tilde{X}} \tilde{\omega}^a_b = \mathcal{L}_{\frac{\partial}{\partial x^1}} \tilde{\omega}^a_b = (y^{-1})^a_r y^s_b \frac{\partial \Gamma_{ks}^r}{\partial x^1} dx^k$$

mathematically equivalent to the representation

$$\mathcal{L}_X \nabla = \frac{\partial \Gamma_{ks}^r}{\partial x^1} dx^k \otimes dx^s \otimes \frac{\partial}{\partial x^r} \quad (\text{A.6})$$

Therefore, under the stated circumstance,  $\mathcal{L}_X \nabla = 0$  if and only if  $\frac{\partial \Gamma_{ks}^r}{\partial x^1} = 0$ .

More generally, in arbitrary coordinates, denoting by  $T^i_{jk}$ ,  $R^i_{jkl}$  and  $\parallel$  respectively the torsion tensor, the curvature tensor and the covariant derivative associated with  $\nabla$ , a straightforward but lengthy calculation yields the result

$$\mathcal{L}_{\tilde{X}} \tilde{\omega}^a_b = (y^{-1})^a_r y^s_b \left[ (X^r_{\parallel s} + X^p T^r_{ps})_{\parallel k} + X^p R^r_{spk} \right] dx^k$$

corresponding to the representation

$$\mathcal{L}_X \nabla = \left[ (X^r_{\parallel s} + X^p T^r_{ps})_{\parallel k} + X^p R^r_{spk} \right] dx^k \otimes dx^s \otimes \frac{\partial}{\partial x^r} \quad (\text{A.7})$$

As an indirect check, the reader may verify that eq. (A.7) reduces to eq. (A.6) whenever the condition  $X = \frac{\partial}{\partial x^1}$  is satisfied. The validity of eq. (A.7) in any coordinate system is then ensured by the tensor character of both sides.

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